

Analyticity Properties of Graham-Witten Anomalies

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ABSTRACT: Analytic properties of Graham-Witten anomalies are considered. Weyl anomalies according to their analytic properties are of type A (coming from δ -singularities in correlators of several energy-momentum tensors) or of type B (originating in counterterms which depend logarithmically on a mass scale). It is argued that all Graham-Witten anomalies can be divided into 2 groups: internal and external, and that all external anomalies are of type B, whereas among internal anomalies there is one term of type A and all the rest are of type B. This argument is checked explicitly for the case of a free scalar field in a 6-dimensional space with a 2-dimensional submanifold.

KEYWORDS: Graham-Witten anomaly, Weyl anomaly.

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1. Introduction

From the beginning of 1970's we know that dilation and conformal invariance of quantum field theories are in general broken by quantum effects. There are two different kinds of breaking: 1) The equations of motion for operators do not coincide with classical ones, i.e. the theory has a nonvanishing β -function; 2) The β -function is 0, so the theory is conformally invariant also at the quantum level, but in certain correlators the corresponding Noether currents possess anomalous divergences. This phenomenon was called “trace (Weyl, conformal) anomaly”. (The first case appears in the theories like QED or QCD. It is described, for example, in [1, 2, 3]. In what follows we deal only with the second case described in [4]).

The problem of identifying and classifying all possible trace anomalies was extensively studied. In [5, 6] this problem was treated by the grading of Weyl variation operator and it was shown that the trace anomalies appear due to a nontrivial cohomology of the Weyl transformations in the space of local diffeoinvariant polynomials. This approach naturally explains why, for example, one sometimes can get rid of an anomaly by changing the procedure of regularizing the ultraviolet divergences, and allows one to identify on general grounds the anomalies that can be disposed of in this way.

An approach to the classification of the trace anomalies taking into account the mechanism producing them was suggested in [7]. It was shown that all local trace anomalies can be divided into two classes (types A and B) with different production mechanisms. Anomalies of type A appear because of the δ -singularities in the discontinuities of correlators of several energy-momentum tensors; the anomaly itself is always equal to the Euler density. On the other hand, anomalies of type B arise because of counterterms that must be added to the effective action in order to make finite certain logarithmically divergent correlators. The corresponding anomalies are given by Weyl invariant polynomials.

The way in which the trace anomalies are reproduced by the gravitational action in the framework of the *AdS/CFT* duality was studied in [8]. In this correspondence one can

compute the generating functional of the conformal field theory on some D -dimensional manifold M by evaluating the gravitational action for the metric on some $D+1$ -dimensional manifold X which has M as its conformal boundary. This gravitational action turns out to have (infrared) divergences due to the integration over the whole $D+1$ -dimensional volume. Covariant regularization of the divergence breaks the Weyl invariance and an anomaly is produced. This is consistent with the fact that conformal field theory on M also has a Weyl anomaly, and this anomaly is reproduced by the anomaly of the regularized gravitational action of X .

This idea can be generalized in a following sense: expectation values of observables which are defined on a submanifold can be anomalous because of the integration over the ambient space. This anomaly (we will call it Graham-Witten, or GW anomaly [9]) is more general than previously known ones since it deals with submanifolds in Riemann or Pseudo-Riemann spaces, whereas usually one considers these spaces themselves. If the submanifold coincides with the whole space then we are in the standard situation. Other cases must be studied separately. In a general situation new terms appear in the expression of the anomaly. These terms are due to the presence of the ambient space.

The GW anomaly can be studied independently of the *AdS/CFT* correspondence. This anomaly was calculated directly in the framework of the quantum field theory for various models in [10, 11]. However, the methods used in the calculations do not reveal the mechanism of the anomaly. In this paper we fulfill this gap by presenting a different method which is in close relation to that of [7]. This method allows one to understand an origin of various terms in the expression of the anomaly. The method was developed in [13].

The paper is organized in a following way. Part 2 describes the classification of Weyl anomalies. Part 3 contains results concerning the GW anomaly, especially those obtained in [10]. In part 4 we consider the general properties of the GW anomaly and present the calculation for massless scalar field in 6 dimensions (this model was treated in a different way in [11]). On this model we determine the production mechanisms of various terms in the expression of the anomaly. Part 5 is the summary of the results.

2. Classification of Weyl anomalies

A classification of the Weyl anomalies according to the mechanism of their appearance was given in [7]. It is based on the observation that the integrated trace anomaly can be considered as a variation of an effective action w.r.t. the mass scale. Then if this integrated anomaly vanishes then the effective action contains no mass scale. This type of the trace anomaly was called in [7] “type A”. If, on the other hand, the integrated anomaly doesn’t vanish then the effective action contains a mass scale and the corresponding anomaly is called “type B”.

As observed in [7] the distinction between the 2 types of anomalies is clear if one uses the dimensional regularization for treating the divergences.

Consider the conformal field theory on a curved D -dimensional Riemann manifold M with some metric $g_{\mu\nu}$. We introduce the effective action $W[g_{\mu\nu}]$ defined as

$$e^{iW[g_{\mu\nu}]} = \int \mathcal{D}\phi e^{iS[g_{\mu\nu}, \phi]}. \quad (2.1)$$

In order to regularize the ultraviolet divergences one takes D to be non-integer but close to some even number $2n$ (we know that there are no anomalies in odd dimensions). Then $W[g_{\mu\nu}]$ becomes finite and no anomalies can appear. However, $W[g_{\mu\nu}]$ can have a pole term at $D = 2n$, meaning that it looks like $W'[g_{\mu\nu}]/\epsilon$, where $\epsilon = D - 2n$. This $1/\epsilon$ term is essentially the source of the anomaly in this approach. There are 2 possibilities: $W'[g_{\mu\nu}]$ may or may not vanish at $D = 2n$. If $W'[g_{\mu\nu}]$ vanishes then the type A anomaly arises, otherwise the corresponding anomaly is type B.

The typical example of the type A situation is the trace anomaly in 2 dimensions which is proportional to the curvature of the space. It is shown in [7] that the effective action for $D \approx 2$ is

$$W_D[g_{\mu\nu}] = -\frac{1}{4} \int_M d^D x \sqrt{g} R \square^{-1-\epsilon/2} R + \frac{1}{\epsilon} \int_M d^D x \sqrt{g} R. \quad (2.2)$$

Here M is the D -dimensional manifold which is obtained from the initial 2-dimensional Riemann space by some kind of an “analytic continuation,” and $g_{\mu\nu}$ and R denote the metric and the curvature of this continued manifold. This effective action is Weyl invariant to the order ϵ in D dimensions.

The second term in this expression has $0/0$ ambiguity since the integral vanishes at $D = 2$. To take the limit is not trivial, since it depends on how M was continued to D dimensions. One can imagine, for example, that the integral in the second term vanishes also for the continued manifold. Then the second term disappears and we are left with a finite expression in 2 dimensions (Polyakov action)

$$W_2[g_{\mu\nu}] = -\frac{1}{4} \int_M d^2 x \sqrt{g} R \square^{-1} R. \quad (2.3)$$

We see that this way of taking the limit is just a kind of regularization of the effective action.

To compute the Weyl variation of W one just notices that in D dimensions $\delta W_D = 0$, so the Weyl variations of two terms in eq.(2.2) cancel. Therefore the Weyl variation of W_2 as we defined it is opposite in sign to that of the dropped local term in (2.2). The latter can be calculated easily, giving

$$\delta W_2 \sim \int_M d^2 x \sqrt{g} R \sigma. \quad (2.4)$$

In the final answer the $1/\epsilon$ cancels, and the result is finite. It is the general feature of the trace anomalies. It was shown in [7] that similar anomalies of type A appear in any even dimension and the corresponding anomaly density is the Euler density.

The simplest example of type B anomaly appears in four dimensions (the so-called *c*-anomaly). As it was shown in [7] the corresponding effective action is

$$W_D[g_{\mu\nu}] = \frac{1}{\epsilon} \int_M d^D x \sqrt{g} C_{\mu\nu\xi\eta} \square^{\epsilon/2} C^{\mu\nu\xi\eta}, \quad (2.5)$$

where $C_{\mu\nu\xi\eta}$ is the Weyl tensor which is like all the other continued to D dimensions. W_D is again Weyl invariant to order ϵ but it diverges at $D = 4$, so one should modify it by adding a suitable counterterm that would produce a finite result. The subtracted W_D is

$$W_D^{sub}[g_{\mu\nu}] = W_D[g_{\mu\nu}] - \frac{\mu^\epsilon}{\epsilon} \int_M d^D x \sqrt{g} C_{\mu\nu\xi\eta} C^{\mu\nu\xi\eta}, \quad (2.6)$$

where μ is some arbitrary mass. Then W_4 can be rewritten as

$$W_4[g_{\mu\nu}] = \frac{1}{2} \int_M d^D x \sqrt{g} C_{\mu\nu\xi\eta} \log\left(\frac{\square}{\mu^2}\right) C^{\mu\nu\xi\eta} + \frac{\mu^\epsilon}{\epsilon} \int_M d^D x \sqrt{g} C_{\mu\nu\xi\eta} C^{\mu\nu\xi\eta}, \quad (2.7)$$

As in the case of the type A anomaly, the effective action is a sum of nonlocal finite term and a pole term. However, the residue in this case doesn't vanish. One of the ways to take the limit $\epsilon \rightarrow 0$ is just to drop the pole term:

$$W_4[g_{\mu\nu}] = \frac{1}{2} \int_M d^D x \sqrt{g} C_{\mu\nu\xi\eta} \log\left(\frac{\square}{\mu^2}\right) C^{\mu\nu\xi\eta}. \quad (2.8)$$

Again, the Weyl variation of the nonlocal term is opposite in sign to that of the pole term, so

$$\delta W_4[g_{\mu\nu}] \sim \int_M d^4 x \sqrt{g} C_{\mu\nu\xi\eta} C^{\mu\nu\xi\eta} \sigma, \quad (2.9)$$

which is finite like in the case of type A.

As stated in the beginning of this section, the essential difference between the 2 types is that the anomalies of type B emerge as a consequence of the mass scale μ that is introduced to the theory by counterterms. On the other hand, the anomalies of type A arise without the scale and reflect the reduction in the number of Weyl invariant expressions when one crosses an even dimension.

In part 4 we will use this classification in order to identify the types of GW anomalies.

3. The Graham-Witten Anomaly

In this part we present the results obtained mostly in [10]. In this paper the GW anomaly was calculated for the free self-dual gauge field on the 6-dimensional Riemann manifold. Here the submanifold for which the GW anomaly should be calculated is just the 2-dimensional surface of the Wilson observable. This observable is dimensionless and, hence, Weyl invariant classically.

So, consider the 6-dimensional Riemann space M and the 2-dimensional submanifold N embedded into M . We denote the coordinates in M as X^μ , $\mu = 1\dots 6$ and the coordinates

in N as ξ^α , $\alpha = 1 \dots 2$. The metric on M is denoted as $G_{\mu\nu}$. The corresponding induced metric on N is $g_{\alpha\beta}$. We also denote as ∇_μ and $\hat{\nabla}_\alpha$ the covariant derivatives in the ambient space along the coordinate line μ and on the submanifold along the coordinate line α correspondingly. We will also use the Christoffel symbols of M and N which we denote as $\Gamma_{\mu\nu}^\lambda$ and $\hat{\Gamma}_{\alpha\beta}^\gamma$. The Riemann tensors, the Ricci tensors and the curvatures of M and N we denote as $R_{\mu\nu\xi\eta}$, $\hat{R}_{\alpha\beta\gamma\delta}$, $R_{\mu\nu}$, $\hat{R}_{\alpha\beta}$, R and \hat{R} .

The gauge field in 6 dimensions is the antisymmetric tensor $A_{\mu\nu}$. The action of the free gauge field is

$$S = -\frac{1}{12} \int_M d^6 X \sqrt{G} F_{\mu\nu\lambda} F^{\mu\nu\lambda}, \quad (3.1)$$

where $F_{\mu\nu\lambda} = 3\nabla_{[\lambda} A_{\mu\nu]}$ is the field strength. The action is invariant under the gauge transformations $A_{\mu\nu} \rightarrow A_{\mu\nu} + \bar{A}_{\mu\nu}$, where $\bar{A}_{\mu\nu}$ is exact form with integer periods.

In what follows we will need the propagator in the Feynmann gauge of this field which we denote as $\Delta_{\mu\nu,\xi\eta}$:

$$\Delta_{\mu\nu,\xi\eta}(X, Y) = \langle 0 | A_{\mu\nu}(X) A_{\xi\eta}(Y) | 0 \rangle. \quad (3.2)$$

It is an antisymmetric bitensor.

Given the 2-dimensional submanifold N , one can construct the Wilson surface observable

$$W[N] = \exp\left(2\pi i \int_N d\xi^\alpha \wedge d\xi^\beta \tilde{A}_{\alpha\beta}\right),$$

$\tilde{A}_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu A_{\mu\nu}$ being the pull-back of $A_{\mu\nu}$ to N . The vacuum expectation value of W is

$$\langle 0 | W[N] | 0 \rangle = e^{-4\pi^2 I},$$

with

$$I = \int_N d\xi_1^\alpha \wedge d\xi_1^\beta \int_N d\xi_2^\gamma \wedge d\xi_2^\delta \tilde{\Delta}_{\alpha\beta,\gamma\delta}(\xi_1, \xi_2), \quad (3.3)$$

where

$$\tilde{\Delta}_{\alpha\beta,\gamma\delta}(\xi_1, \xi_2) = \partial_\alpha X^\mu(\xi_1) \partial_\beta X^\nu(\xi_1) \partial_\gamma X^\xi(\xi_2) \partial_\delta X^\eta(\xi_2) \Delta_{\mu\nu,\xi\eta}(X, Y)$$

is the pull-back of the propagator to the submanifold N . Formally one can say that $\langle 0 | W[N] | 0 \rangle$ is still conformally invariant even after adding the gauge-fixing term to the Lagrangian. But it's not true, however, since I diverges, and the regularization leads to the anomaly.

There are many ways to regularize I without breaking the diffeomorphism invariance. For example, N can be replaced by two 2-dimensional surfaces separated by an infinitesimal distance δ (this regularization is used in [11]). In the regularization used in [10] the points ξ_1 and ξ_2 on N are prevented from being closer to each other than some infinitesimal ϵ . To preserve the diffeomorphism invariance ϵ should be measured by the geodesic distance.

If one adopts this last regularization scheme then, as it was shown in [10], the conformal variation of I is

$$\delta I = \int_N d\xi^1 d\xi^2 \sqrt{g} \left[\frac{4}{\epsilon^2} \sigma - \frac{1}{2} \hat{R} \sigma - \frac{3}{4} \left((\square X)^2 - 4g^{\alpha\beta} \tilde{P}_{\alpha\beta} \right) \sigma - \frac{1}{6} g^{\alpha\gamma} g^{\beta\delta} \tilde{C}_{\alpha\beta\gamma\delta} \sigma - \frac{5}{6} \square X^\mu \nabla_\mu \sigma \right]. \quad (3.4)$$

Here σ is a parameter of the infinitesimal Weyl transformation, $\tilde{C}_{\alpha\beta\gamma\delta}$ is a pull-back of the Weyl tensor of the ambient space $C_{\mu\nu\xi\eta}$, $\tilde{P}_{\alpha\beta}$ is a pull-back of a tensor $P_{\mu\nu}$ which is defined as

$$P_{\mu\nu} = \frac{1}{4} (R_{\mu\nu} - \frac{1}{10} R G_{\mu\nu}), \quad (3.5)$$

$\square = \hat{\nabla}_\alpha \hat{\nabla}^\alpha$ is a Laplacian on N and $\square X^\mu$ is a mean curvature vector which is a trace of a second fundamental form $\Omega_{\alpha\beta}^\mu$:

$$\Omega_{\alpha\beta}^\mu = \partial_\alpha \partial_\beta X^\mu - \hat{\Gamma}_{\alpha\beta}^\delta \partial_\delta X^\mu + \Gamma_{\nu\lambda}^\mu \partial_\alpha X^\nu \partial_\beta X^\lambda \quad (3.6)$$

We see that the conformal variation diverges in the limit $\epsilon \rightarrow 0$. However, one can consider the renormalized Wilson observable

$$W_R = W \int_N d\xi^1 d\xi^2 \left(-\frac{2}{\epsilon^2} \sqrt{g} \right). \quad (3.7)$$

The conformal variation of W_R is finite. It is given by

$$\delta W_R = -4\pi^2 W_R \int_N d\xi^1 d\xi^2 \sqrt{g} \left[-\frac{1}{2} \hat{R} \sigma - \frac{3}{4} \left((\square X)^2 - 4g^{\alpha\beta} \tilde{P}_{\alpha\beta} \right) \sigma - \frac{1}{6} g^{\alpha\gamma} g^{\beta\delta} \tilde{C}_{\alpha\beta\gamma\delta} \sigma - \frac{5}{6} \square X^\mu \nabla_\mu \sigma \right]. \quad (3.8)$$

This expression agrees with the general features of the trace anomaly mentioned above. However, as noticed in [12] the last term represents a trivial anomaly since it is a Weyl variation of a local expression:

$$\int_N d\xi^1 d\xi^2 \sqrt{g} \square X^\mu \nabla_\mu \sigma \propto \delta \int_N d\xi^1 d\xi^2 \sqrt{g} \square X^\mu \square X_\mu,$$

and therefore the true anomaly is given by just 3 first terms.

In the next part we are going to examine this result in detail and, in particular, answer the following question: What type is this anomaly?

4. Analysis of GW anomaly

In this part we present a general analysis of the GW anomalies and give a detailed calculation in the simplest case of 2-dimensional surface embedded into a 6-dimensional space.

It is well-known that the trace anomaly can appear only in spaces of even dimension. It follows, for example, from the cohomological analysis of [5, 6]. In the case of the GW anomaly the precise connection between the dimension D of the ambient space and the dimension d of the surface of an observable is

$$d = \frac{D-2}{2} \quad (4.1)$$

For odd-dimensional spaces (odd D) the dimensions of the “conformally invariant” submanifolds is to be half-integer, which is impossible. This result essentially shows that for odd D one cannot construct the self-dual gauge field theory. Such a field is a scalar in 2 dimensions, a vector in 4 dimensions, an antisymmetric tensor of the rank 2 in 6 dimensions etc. There is no room for odd dimensions in this scheme. The same situation occurs if we try to construct nonlocal variables from scalars as we’ll discuss in the following.

In order to proceed we will need to make a few general observations about the conformal variations on submanifolds. Suppose that we make the conformal variation of the metric in the ambient space M . The change in the metric is

$$\delta G_{\mu\nu} = 2\sigma G_{\mu\nu}. \quad (4.2)$$

Together with this the induced metric on the submanifold N $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$ undergoes changes as well. Since the derivatives of the coordinates are invariant under the conformal transformations, $g_{\alpha\beta}$ under (4.2) changes in way similar to $G_{\mu\nu}$

$$\delta g_{\alpha\beta} = 2\sigma g_{\alpha\beta}. \quad (4.3)$$

Therefore all the combinations of $g_{\alpha\beta}$ and its derivatives, $\hat{R}_{\mu\nu\xi\eta}$ for example, change under the transformation as if there were no ambient space at all. Weyl anomalies are local diffeoinvariant expressions constructed of “geometrical objects” of a definite dimension and linear in σ . Therefore we can say that among all possible GW anomalies there are always those that “belong” just to the submanifold N . These are the usual trace anomalies in the space of dimension d . Notice that they exist only for even d .

If there were no ambient space M then these would be the only possible anomalies. But the presence of M allows for new local diffeoinvariant expressions which are built from the terms among which there are some that don’t “belong” to N . We will call the anomalies that belong to N “internal” and all other “external”.

Consider now the local terms that don’t belong to N . They can involve, for example, the Riemann tensor, the Ricci tensor and the curvature of M , the second fundamental form and their covariant derivatives. One can easily check that any such expression with all the indices contracted is of even mass dimension. Therefore the GW anomaly apparently is possible only if N is even dimensional. For the free self-dual gauge field it can occur according to (4.1) only if

$$D = 2 + 4n, \quad n = 0, 1, 2, \dots \quad (4.4)$$

The case $D = 2$ is trivial since N is of dimension 0 and is just a set of points. The first non-trivial case is $D = 6$ which is dealt with in [10].

We are now going to show that all external anomalies are of type B. We saw in part 2 that the anomalies of type B arise as conformal variations of pole terms with non-vanishing residue. What we will argue is that any pole term that produces an anomaly generically doesn't vanish. To do so notice first that no external anomaly can be a conformal variation of an internal expression since any such variation is also internal. Then suppose that some external anomaly is a conformal variation of a pole term with vanishing residue. This residue is an integral over N of an expression which contains some "external" quantities like the second fundamental form, for example. Then we can change M at least in some small domain in such a way that N (and therefore all internal quantities) will remain the same but some external quantities will change. Then the value of the integral will change. So we see that the residue doesn't generically vanish.

Similar situation occurs with the usual type B anomalies. Consider again the example of type B anomaly given in part 2. One can imagine that for some particular manifold the integral in the second term in the eq. (2.7) vanishes. But it doesn't mean that the corresponding anomaly is of type A since we are to consider generic manifolds.

Consider the case $D = 6$ and $d = 2$ in more detail. In order to do this we instead of δI as was done in [10, 11] (see eq. (3.4)) calculate I itself (the calculation was carried out in [13]). It turns out that for our purposes one doesn't really have to work with the gauge field. This is so because the only essential features of the theory that are necessary for us are its symmetries. There are 3 different symmetries here: 1) symmetry with respect to the diffeomorphisms of M , 2) symmetry with respect to the diffeomorphisms of N , 3) Weyl symmetry in M (and therefore in N). Presence of the GW anomaly tells us that no regularization can preserve all of them, and this fact is independent of any particular theory. Therefore we can choose any theory we like and do all the calculations with it. We choose the simplest possible theory, the free massless scalar field theory. This choice was suggested in [11, 13]. But instead of the point-splitting regularization of [10] and [11] we implement the dimensional regularization for the reasons explained above.

The action of free massless scalar field ϕ in D -dimensional curved space M is given by

$$S = \int_M d^D x \sqrt{G} (G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - A R \phi^2), \quad (4.5)$$

where

$$A = \frac{D-2}{4(D-1)}. \quad (4.6)$$

With the usual Weyl transformation law for the scalar field $\delta\phi = -\frac{D-2}{2}\sigma\phi$ this action is Weyl invariant (our convention for the curvature is $R_{\mu\rho\nu\sigma} = G_{\sigma\xi}(\partial_\mu\Gamma_{\rho\nu}^\xi - \partial_\rho\Gamma_{\mu\nu}^\xi) + \dots$).

For any $\frac{D-2}{2}$ - dimensional submanifold N we, following [11], construct the analog of the Wilson loop

$$W[N] = \exp\left(2\pi i \int_N d^{\frac{D-2}{2}} \xi \sqrt{g(\xi)} \phi(\xi)\right). \quad (4.7)$$

Then we can evaluate its vacuum expectation value

$$\langle 0|W[N]|0 \rangle = \int \mathcal{D}\phi W[N] e^{-S}. \quad (4.8)$$

This path integral can be calculated exactly and the result is

$$\langle 0|W[N]|0 \rangle = e^{-4\pi^2 I}, \quad (4.9)$$

where I is given now by

$$I = \int_N d^{\frac{D-2}{2}} \xi_1 \sqrt{g(\xi_1)} \int_N d^{\frac{D-2}{2}} \xi_2 \sqrt{g(\xi_2)} \Delta(\xi_1, \xi_2). \quad (4.10)$$

Δ here is the propagator of the scalar field. Consider 2 nearby points in M , which we denote by X_0 and X . We introduce the Riemann normal coordinates X^μ on M at X_0 . In these coordinates the short-distance expansion of the propagator between X_0 and X is

$$\Delta(X, X_0) = \frac{\alpha}{|X|^{D-2}} \left(1 - \frac{D-2}{12} P_{\mu\nu} X^\mu X^\nu + \dots \right). \quad (4.11)$$

Here

$$P_{\mu\nu} = \frac{1}{D-2} \left(R_{\mu\nu} - \frac{R}{2(D-1)} G_{\mu\nu} \right), \quad (4.12)$$

where $R_{\mu\nu}$ and R are the Ricci tensor and the curvature scalar of M at X_0 (for $D = 6$ it coincides with the tensor defined in (3.5)). This tensor possesses a very simple conformal variation

$$\delta P_{\mu\nu} = \nabla_\mu \nabla_\nu \sigma. \quad (4.13)$$

The coefficient α in (4.11) is given by

$$\alpha = -\frac{1}{(D-2)\sigma_D}, \quad (4.14)$$

with $\sigma_D = 2\pi^{D/2}/\Gamma(\frac{D}{2})$ being an area of a unit sphere in D -dimensional flat Euclidean space.

In order to calculate the propagator between 2 points on the submanifold N we need the expression for $|X|$ in the coordinates ξ on N . For this purpose consider 2 nearby points A_0 and A on N . At A_0 we introduce normal coordinates on N that we denote by u^α (in addition to the previously introduced normal coordinates in M). We consider the difference of the X -coordinates of A and A_0 :

$$X^\mu(A) = X^\mu(A_0) + \partial_\alpha X^\mu(A_0) u^\alpha + \frac{1}{2} \partial_\alpha \partial_\beta X^\mu(A_0) u^\alpha u^\beta + \frac{1}{6} \partial_\alpha \partial_\beta \partial_\gamma X^\mu(A_0) u^\alpha u^\beta u^\gamma + \dots \quad (4.15)$$

For $|X(A) - X(A_0)|^2$ we have

$$\begin{aligned} |X(A) - X(A_0)|^2 &= \partial_\alpha X^\mu \partial_\beta X_\mu u^\alpha u^\beta + \partial_\alpha \partial_\beta X^\mu \partial_\gamma X_\mu u^\alpha u^\beta u^\gamma + \\ &\quad + \left(\frac{1}{4} \partial_\alpha \partial_\beta X^\mu \partial_\gamma \partial_\delta X_\mu + \frac{1}{3} \partial_\alpha \partial_\beta \partial_\gamma X^\mu \partial_\delta X_\mu \right) u^\alpha u^\beta u^\gamma u^\delta + \dots \end{aligned}$$

In the first term $\partial_\alpha X^\mu \partial_\beta X_\mu$ is the induced metric at A_0 which is $\delta_{\alpha\beta}$. The second term vanishes because in the normal coordinates $\partial_\alpha \partial_\beta X^\mu$ is orthogonal to N . We use this last fact to get

$$\partial_\alpha \partial_\beta \partial_\gamma X^\mu \partial_\delta X_\mu = \partial_\alpha \left(\partial_\beta \partial_\gamma X^\mu \partial_\delta X_\mu \right) - \partial_\alpha \partial_\beta X^\mu \partial_\gamma \partial_\delta X_\mu = -\partial_\alpha \partial_\beta X^\mu \partial_\gamma \partial_\delta X_\mu \quad (4.16)$$

and then for $|X(A) - X(A_0)|^2$ we obtain

$$|X(A) - X(A_0)|^2 = u^2 - \frac{1}{12} \delta_{\mu\nu} \Omega_{\alpha\beta}^\mu \Omega_{\gamma\delta}^\nu u^\alpha u^\beta u^\gamma u^\delta + \dots, \quad (4.17)$$

since in the normal coordinates $\partial_\alpha \partial_\beta X^\mu$ coincides with the second fundamental form $\Omega_{\alpha\beta}^\mu$ defined in (3.6).

We come back now to the calculation of I in eq. (4.10). We are interested essentially in the pole of I , as explained above. Such a pole arises when the coordinates ξ_1 and ξ_2 come close to each other. Therefore we may take the internal integral in (4.10) just over a small neighborhood of the point ξ_1 . The particular form of this neighborhood is unimportant, so we take it to be the circle of some small “geodesic radius” r . We denote this circle by C_r . Later we will impose some constraints on r , but the final answer will be independent of r .

Consider the internal integral in (4.10)

$$J(\xi_1) = \int_{C_r} d^{\frac{D-2}{2}} \xi_2 \sqrt{g} \Delta(\xi_1, \xi_2).$$

We introduce the normal coordinates at ξ_1 both in M and in N . Using (4.17) we write the propagator in these coordinates

$$\Delta(u, 0) = \alpha \frac{1 - \frac{D-2}{12} \tilde{P}_{\alpha\beta} u^\alpha u^\beta + \dots}{\left(u^2 - \frac{1}{12} \delta_{\mu\nu} \Omega_{\alpha\beta}^\mu \Omega_{\gamma\delta}^\nu u^\alpha u^\beta u^\gamma u^\delta + \dots \right)^{\frac{D-2}{2}}}.$$

The notation here and in what follows reproduces that of section 3: a tilde denotes a pull-back of a corresponding tensor from M to N , all internal quantities are denoted by a hat. The expansion of \sqrt{g} in normal coordinates is

$$\sqrt{g} = 1 + \frac{1}{6} \hat{R}_{\alpha\beta} u^\alpha u^\beta + \dots$$

In our coordinate system J is taken at the origin and is given by

$$J(0) = \alpha \int_{C_r} \frac{1 + \frac{1}{6} \hat{R}_{\alpha\beta} u^\alpha u^\beta - \frac{D-2}{12} \tilde{P}_{\alpha\beta} u^\alpha u^\beta + \dots}{(u^2)^{\frac{D-2}{2}} \left(1 - \frac{1}{12u^2} \delta_{\mu\nu} \Omega_{\alpha\beta}^\mu \Omega_{\gamma\delta}^\nu u^\alpha u^\beta u^\gamma u^\delta + \dots \right)^{\frac{D-2}{2}}} d^{\frac{D-2}{2}} u. \quad (4.18)$$

From this expression we see that r should be small enough so that the denominator in the integrand wouldn't vanish.

Now we expand the integrand in powers of u . Since the integration goes over a circle we can introduce polar coordinates. Then we can replace $u^\alpha u^\beta$ by $u^2 \delta_{\alpha\beta}/d$ etc. and then turn back to the denominator of the form of (4.18). The integral obtained in this way is not equal to J , but differs from it only by finite terms, so the difference is unimportant for us. We get

$$J(0) = \alpha \int_{C_r} \frac{(u^2)^{-\frac{D-2}{2}} \left(1 + \frac{\hat{R}}{3(D-2)} u^2 - \frac{1}{6} P u^2 + \dots \right) d^{\frac{D-2}{2}} u}{\left[1 - \frac{u^2}{3(D-2)(D+2)} \delta_{\mu\nu} \Omega_{\alpha\beta}^\mu \Omega_{\gamma\delta}^\nu (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \right]^{\frac{D-2}{2}}} + \text{finite}, \quad (4.19)$$

where $P = g^{\alpha\beta} \tilde{P}_{\alpha\beta}$. We use the following notation

$$K = \frac{\hat{R}}{3(D-2)} - \frac{P}{6}, \quad (4.20)$$

$$L = \frac{1}{3(D-2)(D+2)} \delta_{\mu\nu} \Omega_{\alpha\beta}^{\mu} \Omega_{\gamma\delta}^{\nu} (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}). \quad (4.21)$$

The angular integration is easily performed. The result is

$$J(0) = \alpha \sigma_{\frac{D-2}{2}} \int_0^r \frac{(1 + K u^2) u^{\frac{D-2}{2}-1}}{(u^2)^{\frac{D-2}{2}} (1 - L u^2)^{\frac{D-2}{2}}} du, \quad (4.22)$$

where like in the eq. (4.14) σ_d means the area of a unit sphere in the d -dimensional Euclidean space. By changing the integration variable the last integral can be transformed to the form

$$J(0) = \alpha \sigma_{\frac{D-2}{2}} r^{1-D/2} \int_0^1 \frac{(t^{-D/4-1/2} + K t^{-D/4+1/2})}{(1 - L r^2 t)^{\frac{D-2}{2}}} dt.$$

This integral can be evaluated using the following identity:

$$\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b, c; z), \quad (4.23)$$

where ${}_2F_1(a, b, c; z)$ is a hypergeometric function defined as

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}. \quad (4.24)$$

The use of these identities gives

$$J(0) = r^{1-D/2} \frac{\alpha \sigma_{\frac{D-2}{2}}}{\Gamma(\frac{D-2}{2})} \sum_{n=0}^{\infty} \Gamma\left(\frac{D-2}{2} + n\right) \frac{(r^2 L)^n}{n!} \left[\frac{1}{-\frac{D}{4} + \frac{1}{2} + n} + \frac{r^2 K}{-\frac{D}{4} + \frac{3}{2} + n} \right]. \quad (4.25)$$

Γ -functions don't produce poles, so the only poles are those that come from the denominators in brackets. At $D \approx 6$ there are poles at $n = 0, 1$. So the divergent part of J is

$$J(0) = \alpha \sigma_{\frac{D-2}{2}} r^{3-D/2} \frac{1}{\frac{3}{2} - \frac{D}{4}} \left(K + \frac{D-2}{2} L \right). \quad (4.26)$$

We see that at $D = 6$ r disappears as we expected.

We will now bring L into the covariant form. We defined it in eq. (4.21) as

$$L = \frac{1}{3(D-2)(D+2)} \delta_{\mu\nu} \Omega_{\alpha\beta}^{\mu} \Omega_{\gamma\delta}^{\nu} (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}). \quad (4.27)$$

There are 2 kinds of terms here. In order to calculate $\delta_{\mu\nu} \delta_{\alpha\beta} \delta_{\gamma\delta} \Omega_{\alpha\beta}^{\mu} \Omega_{\gamma\delta}^{\nu}$ we notice that in the normal coordinates $g^{\alpha\beta} \Omega_{\alpha\beta}^{\mu} \equiv \square X^{\mu}$, where $\square X^{\mu}$ is a mean curvature vector, and

therefore this term is $\square X^\mu \square X_\mu$. In order to “covariantize” other terms in (4.27) we use the Gauss-Codazzi equation

$$\hat{R}_{\alpha\beta\gamma\delta} = \tilde{R}_{\alpha\beta\gamma\delta} + G_{\mu\nu}(\Omega_{\alpha\delta}^\mu \Omega_{\beta\gamma}^\nu - \Omega_{\alpha\gamma}^\mu \Omega_{\beta\delta}^\nu), \quad (4.28)$$

which can be quickly verified in the normal coordinates. Here, as usual, $\hat{R}_{\alpha\beta\gamma\delta}$ is the Riemann tensor of N and $\tilde{R}_{\alpha\beta\gamma\delta}$ is the pull-back of the Riemann tensor of M to N . Contracting the eq. (4.28) with $\delta_{\alpha\beta}\delta_{\gamma\delta}$, we get

$$\delta_{\mu\nu}\delta_{\alpha\gamma}\delta_{\beta\delta}\Omega_{\alpha\beta}^\mu\Omega_{\gamma\delta}^\nu = \square X^\mu \square X_\mu + \hat{R} - \delta_{\alpha\gamma}\delta_{\beta\delta}\tilde{C}_{\alpha\beta\gamma\delta} - (D-4)P. \quad (4.29)$$

We denote $\delta_{\alpha\gamma}\delta_{\beta\delta}\tilde{C}_{\alpha\beta\gamma\delta}$ as C . Plugging all these into the eq. (4.27) gives

$$L = \frac{1}{3(D-2)(D+2)} \left(3\square X^\mu \square X_\mu + 2\hat{R} - 2C - 4P \right).$$

Finally, the pole part of the internal integral is

$$J(0) = 2\alpha\sigma_{\frac{D-2}{2}}r^{3-D/2} \frac{1}{3-D/2} \left[\frac{2D}{3(D-2)(D+2)} \hat{R} - \frac{1}{3(D-2)} C + \frac{2}{D+2} \square X^\mu \square X_\mu - \left(\frac{1}{6} + \frac{2}{3(D+2)} \right) P \right].$$

Everywhere except the pole we can put $D = 6$. Then

$$J(0) = 2\alpha\sigma_2 \frac{1}{3-D/2} \left(\frac{1}{8} \hat{R} - \frac{1}{24} C + \frac{1}{16} (\square X^\mu \square X_\mu - 4P) \right).$$

We calculated J in the normal coordinates. But the result is fully covariant and therefore independent of any particular coordinate frame. Having calculated J , we can finally write down the pole part of I :

$$I_{pole} = \frac{\sigma_2}{\sigma_6} \frac{1}{D-6} \int_N \sqrt{g} \left(\frac{1}{8} \hat{R} - \frac{1}{24} C + \frac{1}{16} (\square X^\mu \square X_\mu - 4P) \right) d^2\xi. \quad (4.30)$$

Here we substituted the precise expression for α from (4.14).

The conformal variations of various terms of the last expression are

$$\begin{aligned} \delta(\sqrt{g} \hat{R}) &= \frac{D-6}{2} \sqrt{g} \hat{R} \sigma + (D-3) \sqrt{g} \square \sigma, \\ \delta(\sqrt{g} C) &= \frac{D-6}{2} \sqrt{g} C \sigma, \\ \delta(\sqrt{g} \square X^\mu \square X_\mu) &= \frac{D-6}{2} \sqrt{g} \square X^\mu \square X_\mu \sigma - (D-2) \sqrt{g} \square X^\mu \partial_\mu \sigma, \\ \delta(\sqrt{g} P) &= \frac{D-6}{2} \sqrt{g} P \sigma + \sqrt{g} \square \sigma - \sqrt{g} \square X^\mu \partial_\mu \sigma. \end{aligned}$$

All the terms proportional to $\square \sigma$ vanish when integrated over N . Therefore the conformal variation of the pole of I is

$$\delta I_{pole} = \frac{1}{8\pi^2} \int_N d^2\xi \sqrt{g} \left[\hat{R} \sigma - \frac{1}{3} C \sigma + \frac{1}{2} (\square X^\mu \square X_\mu - 4P) \sigma - \square X^\mu \partial_\mu \sigma \right]. \quad (4.31)$$

The pole cancels as expected. This expression reproduces the result of [11] for the considered model and is in a complete agreement with the results of [10] discussed in the previous part (in a sense that there appear the same terms in the expression for the anomaly). However, as discussed after eq. (3.8) the last term represents a trivial anomaly, and therefore only 3 first terms are relevant:

$$\delta I_{pole} = \frac{1}{8\pi^2} \int_N d^2\xi \sqrt{g} \left[\hat{R}\sigma - \frac{1}{3}C\sigma + \frac{1}{2}(\square X^\mu \square X_\mu - 4P)\sigma \right]. \quad (4.32)$$

Now we can understand the types of various anomalies. The first term which is proportional to $\hat{R}\sigma$ appeared as the conformal variation of \hat{R} and is of the type A (see the eqs. (2.2)-(2.4)). Two other terms are of the type B since they emerge as conformal variations of nonvanishing expressions.

5. Conclusion

In this paper we have considered the general features of the GW anomaly. It apparently occurs in the spaces of dimension $D = 2 + 4n$, $n = 0, 1, 2$ etc. We showed that in general the GW anomaly consists of 2 groups of terms: internal anomalies that are the usual trace anomalies on the submanifold and external anomalies that are due to the ambient space. We argued also that all the external anomalies were of type B.

For the lowest possible dimension 6 we found all possible expressions for the anomaly and investigated their types using the model of free massless scalar field coupled minimally to the gravity. We found that there were 2 external anomalies of type B and a single internal anomaly of type A.

Using these results we can make a few further guesses about general features of the GW anomalies. It seems natural to guess that the internal anomalies will always behave as described in [7], i.e. among them there will always be a single type A anomaly proportional to the Euler density and a few anomalies of type B. In addition there will be also a few external anomalies of type B. Then for the 10-dimensional case (the next non-trivial one) with the 4-dimensional submanifold N we may guess that there are 2 internal anomalies, namely

$$\begin{aligned} \Delta_1^{int} &= \int_N d^4\xi \sqrt{g} (\hat{R}_{\alpha\beta\gamma\delta} \hat{R}^{\alpha\beta\gamma\delta} - 4\hat{R}_{\alpha\beta} \hat{R}^{\alpha\beta} + \hat{R}^2) \sigma, \\ \Delta_2^{int} &= \int_N d^4\xi \sqrt{g} \hat{C}_{\alpha\beta\gamma\delta} \hat{C}^{\alpha\beta\gamma\delta} \sigma \end{aligned}$$

in the obvious notation. Here Δ_1 is proportional to the Euler density and hence is of type A and Δ_2 is of type B. In addition we expect a few additional external terms in the anomaly. Our prediction is that all these are of type B. It is interesting to check this prediction by an explicit calculation.

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